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Application of Invariant Imbedding to the Buckling of Columns

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ABSTRACT

The classical eigenvalue problem is transformed into an initial value problem by means of invariant imbedding. This initial value form is eminently suited for numerical integration. Several examples are presented to illustrate the technique and to demonstrate the accuracy of the numerical solution.

I. INTRODUCTION

A problem of importance in many branches of engineering and physics is that of determining the eigenvalues of systems of ordinary differential equations. Some traditional techniques of determining eigenvalues are finding the analytical solution to the equations or using an approximate method such as that of Ritz. The analytic solution method can be utilized only for the rather restricted class of problems in which the solution of the equations can be written in terms of elementary functions.

The purpose of this paper is to show how the method of invariant imbedding can be used to convert eigenvalue problems into initial value problems which can be readily integrated on a digital computer. The use of invariant imbedding in the determination of eigenvalues in mechanics was first suggested by Kalaba [1]. A later paper by Shoemaker [2] applied the method of invariant imbedding to the determination of the eigenvalues of second order equations.

In a previous paper [3] the authors have shown how invariant imbedding can be applied to the study of the deflection of thin beams. The eigenvalue problems treated in this paper are taken from the field of the buckling of columns; however, the method is applicable to eigenvalue problems in general.

The presentation is designed to be self-contained, and no previous knowledge of invariant imbedding is assumed.

II. System Equations

In order to illustrate the technique of invariant imbedding, the following system is considered. A column with stiffness EI and length L_1 is loaded by an axial compressive force of magnitude P (see Fig. 1). The column is supported laterally

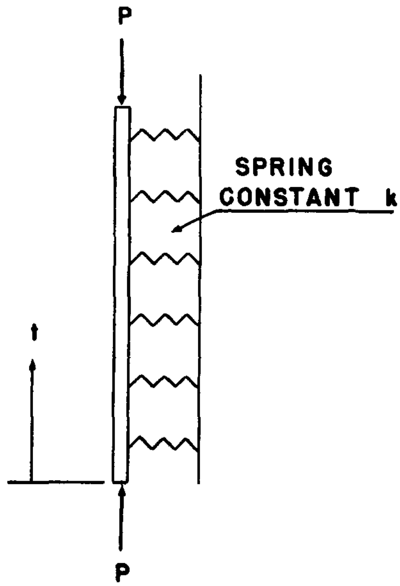

FIG. 1. The physical situation.

by an elastic foundation of spring constant k. The spring constant and the stiffness can be functions of the position along the column. Various types of end supports are considered.

From elementary structural mechanics, it is seen that equilibrium of the column is expressed by the fourth-order ordinary differential equation

$$\frac{d^2}{dt^2}\left(EI\frac{d^2u}{dt^2}\right) + P\frac{d^2u}{dt^2} + ku = 0, \tag{1}$$

where u is the transverse displacement of the neutral axis. The various types of boundary conditions to be imposed are shown in Eq. (2):

$$\begin{aligned} &\text{Simple Support} \quad u = 0, && d^2u/dt^2 = 0,\\ &\text{Clamped Support} \quad u = 0, && du/dt = 0,\\ &\text{Free} \quad \frac{d^2u}{dt^2} = 0, && \frac{d}{dt}\left(EI\frac{d^2u}{dt^2}\right) + P\frac{du}{dt} = 0. \end{aligned} \tag{2}$$

III. Invariant Imbedding Equations

As previously mentioned, the goal of invariant imbedding is to convert the two-point boundary-value problem to an initial-value problem. In order to accomplish this goal, the original problem is imbedded in a larger class of problems, where the length of the column and the boundary conditions at one end are the parameters that describe the members of this larger class.

It is characteristic of invariant imbedding that the equations must be derived for each set of boundary conditions used. In this section we shall derive the invariant imbedding equations for several different types of support.

Example 1: *Simply Supported Column.*

Consider a uniform column which is simply supported at both ends. The governing equation and boundary conditions can then be expressed as

$$\begin{gathered} \mathscr{L}\{u\} = u'''' + \frac{P}{EI} u'' + \frac{k}{EI} u = 0, \\ u(0) = 0, \qquad u(L_1) = a, \qquad u''(0) = 0, \qquad u''(L_1) = b. \end{gathered} \tag{3}$$

Note that the solution to the desired problem is obtained when $a = b = 0$.

Because of the linearity of Eq. (3) we can write

$$u = aw(t, L) + bz(t, L), \tag{4}$$

where the functions w and z satisfy the problems

$$\begin{gathered} \mathscr{L}\{w\} = 0, \qquad \mathscr{L}\{z\} = 0, \\ w(0, L) = 0, \qquad z(0, L) = 0, \\ w''(0, L) = 0, \qquad z''(0, L) = 0, \\ w(L, L) = 1, \qquad z(L, L) = 0, \\ w''(L, L) = 0, \qquad z''(L, L) = 1, \\ 0 \leqslant L \leqslant L_1 . \end{gathered} \tag{5}$$

We begin our derivation of the invariant imbedding equations by differentiating Eq. (5) with respect to the length of column L. A prime sign will denote differentiation with respect to t, and a subscript 2 will denote differentiation with respect to the second parameter L when necessary.

Keeping Eq. (2) in mind, we obtain the formulas

$$\begin{aligned} \mathscr{L}\{w_L\} &= 0, & \mathscr{L}\{z_L\} &= 0, \\ w_L(0, L) &= 0, & z_L(0, L) &= 0, \\ w_L''(0, L) &= 0, & z_L''(0, L) &= 0, \\ w'(L, L) + w_2(L, L) &= 0, & z'(L, L) + z_2(L, L) &= 0, \\ w'''(L, L) + w_2''(L, L) &= 0, & z'''(L, L) + z_2''(L, L) &= 0. \end{aligned} \tag{6}$$

From Eq. (6) we write

$$\begin{aligned} w_2(L, L) &= -w'(L, L), & z_2(L, L) &= -z'(L, L), \\ w_2''(L, L) &= -w'''(L, L), & z_2''(L, L) &= -z'''(L, L). \end{aligned} \tag{7}$$

Since the variables w, z, w_2, and z_2 satisfy the same linear differential equation, we may utilize the superposition principle and the boundary conditions (7) to write

$$w_2(t, L) = -w'(L, L)\, w(t, L) - w'''(L, L)\, z(t, L), \tag{8a}$$

$$z_2(t, L) = -z'(L, L)\, w(t, L) - z'''(L, L)\, z(t, L). \tag{8b}$$

Let us now introduce the auxiliary dependent variables

$$n(L) = w'(L, L), \tag{9a}$$

$$p(L) = z'(L, L), \tag{9b}$$

$$r(L) = w'''(L, L), \tag{9c}$$

$$s(L) = z'''(L, L). \tag{9d}$$

Note that the notation and derivation for this example are similar to those of [3]. Differentiating Eq. (9a) with respect to L, we obtain

$$n' = w''(L, L) + w_2'(L, L). \tag{10}$$

On differentiating Eq. (8a) with respect to t, we obtain the equation

$$w_2'(t, L) = -w'(L, L)\, w'(t, L) - w'''(L, L)\, z'(t, L), \tag{11}$$

or, when $t = L$,

$$w_2'(L, L) = -w'(L, L)\, w'(L, L) - w'''(L, L)\, z'(L, L). \tag{12}$$

From Eq. (9) we see that

$$w_2'(L, L) = -n^2 - rp. \tag{13}$$

From the boundary conditions of Eq. (5) it is seen that

$$w_2''(L, L) = 0.$$

Thus, we may write

$$n' = -n^2 - rp. \tag{14}$$

Similarly for p it is observed that

$$p'(L) = z''(L, L) + z_2'(L, L). \tag{15}$$

But

$$z''(L, L) = 1,$$

and from Eq. (8)

$$z_2'(L, L) = -pn - sp, \tag{16}$$

so that

$$p' = 1 - p(n + s). \tag{17}$$

Differentiating Eq. (9c) yields

$$r'(L) = w''''(L, L) + w_2'''(L, L). \tag{18}$$

Now from Eq. (5) we have

$$w''''(L, L) = -\frac{P}{EI} w''(L, L) - \frac{k}{EI} w(L, L) = -\frac{k}{EI}, \tag{19}$$

and from Eq. (8)

$$\begin{aligned} w_2'''(L, L) &= -w'(L, L)\, w'''(L, L) - w'''(L, L)\, z'''(L, L) \\ &= -nr - rs. \end{aligned} \tag{20}$$

Thus, it is seen that

$$r' = -\frac{k}{EI} - r(n + s). \tag{21}$$

Similarly we have

$$s' = z''''(L, L) + z_2'''(L, L), \tag{22}$$

$$z''''(L, L) = -\frac{P}{EI} z''(L, L) - \frac{k}{EI} z(L, L) = -\frac{P}{EI}, \tag{23}$$

and

$$\begin{aligned} z_2'''(L, L) &= -z'(L, L)\, w'''(L, L) - z'''(L, L)\, z'''(L, L) \\ &= -pr - s^2, \end{aligned} \tag{24}$$

so that

$$s' = -\frac{P}{EI} - pr - s^2. \tag{25}$$

In order to determine the initial conditions, observe that

$$w(0, L) = 0, \qquad w(L, L) = 1.$$

If we consider the limiting case as $L \to 0$, we see that

$$\lim_{L \to 0} \frac{dw}{dL} = +\infty,$$

so that

$$n(0) = +\infty. \tag{26}$$

Also, since

$$w''(0, L) = 0, \qquad w''(L, L) = 0,$$

taking the limit as $L \to 0$ we see that

$$\lim_{L \to 0} \frac{dw''}{dL} = 0,$$

or

$$r(0) = 0. \tag{27}$$

By similar arguments it is found that

$$\begin{aligned} p(0) &= 0, \\ s(0) &= +\infty. \end{aligned} \tag{28}$$

The invariant imbedding formulation of this example is now complete. The governing equations and initial conditions are summarized as follows:

$$\begin{aligned} n' &= -(n^2 + rp), \\ p' &= 1 - p(n + s), \\ r' &= -\frac{k}{EI} \not- r(\quad + s), \\ s' &= -\frac{P}{EI} - (pr + s^2), \end{aligned} \tag{29}$$

$$\begin{aligned} n(0) &= +\infty, \qquad & r(0) &= 0, \\ p(0) &= 0, \qquad & s(0) &= +\infty. \end{aligned} \tag{30}$$

Methods for handling the infinite initial conditions are given below.

Example 2: Free/Clamped Column.

For the second example, we choose a column which is free at the end $t = 0$ and clamped at the end $t = L_1$. In order to illustrate how matrix notation can be used to advantage in the invariant imbedding formulation, this example will be derived in that way. Lower-case symbols are used to denote scalars and vectors, and upper-case symbols are used to represent matrices (exceptions being the length L, the load P, and the stiffness EI). Introducing the notation

$$u(t) = \begin{pmatrix} EIu''(t) \\ EIu'''(t) + Pu'(t) \end{pmatrix}, \qquad v(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}, \tag{31}$$

Eq. (1) can be written as

$$\begin{aligned} du/dt &= Au + Bv, \\ -dv/dt &= Cu + Dv, \end{aligned} \tag{32}$$

where

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ -1/EI & 0 \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & -P \\ -k & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{33}$$

The associated boundary conditions are

$$u(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad v(L_1) = c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{34}$$

The derivation of the invariant imbedding equations follows the lines of the derivation in the previous example. We begin by imbedding the problem in a larger class of problems, where the parameters L and c describe the members of this larger class. Using the linearity of Eq. (32) we write

$$\begin{aligned} v(t) &= V(t, L)\, c, \\ u(t) &= U(t, L)\, c, \end{aligned} \tag{35}$$

where, for the 2×2 matrices U and V, we have

$$\begin{aligned} V(L, L) &= I \qquad \text{Identity Matrix}, \\ U(0, L) &= 0 \qquad \text{Null Matrix}, \end{aligned} \tag{36}$$

and

$$dU/dt = AU + BV,$$
$$-dV/dt = CU + DV. \tag{37}$$

As before, we shall use a prime sign to denote differentiation with respect to t and a subscript 2 to denote differentiation with respect to the second parameter L when needed.

Differentiate Eqs. (37) with respect to L to obtain the relations

$$(U_L)' = AU_L + BV_L, \tag{38}$$

$$-(V_L)' = CU_L + DV_L, \qquad 0 \leqslant t \leqslant L. \tag{39}$$

We also see that

$$U_L(0, L) = 0, \tag{40}$$

and

$$V'(L, L) + V_2(L, L) = 0. \tag{41}$$

It follows that

$$U_L(t, L) = U(t, L)[-V'(L, L)], \tag{42}$$

and

$$V_L(t, L) = V(t, L)[-V'(L, L)], \qquad L \geqslant t. \tag{43}$$

But from Eqs. (37) we obtain the formula

$$-V'(L, L) = CU(L, L) + DV(L, L). \tag{44}$$

By referring to Eq. (36) and introducing the matrix R to be

$$R(L) = U(L, L), \tag{45}$$

it is seen that

$$-V'(L, L) = CR(L) + D, \qquad L \geqslant 0. \tag{46}$$

It follows that the equations for U and V are

$$U_L(t, L) = U(t, L)[CR(L) + D], \tag{47}$$

and

$$V_L(t, L) = V(t, L)[CR(L) + D], \qquad L \geqslant t. \tag{48}$$

The initial conditions at $L = t$ are

$$U(t, t) = R(t) \tag{49}$$

and

$$V(t, t) = I. \tag{50}$$

The function R must now be considered. Differentiation of Eq. (45) shows that

$$\begin{aligned} R'(L) &= U'(L, L) + U_2(L, L) \\ &= AU(L, L) + BV(L, L) + U(L, L)[CR(L) + D]. \end{aligned} \tag{51}$$

It follows that the Cauchy problem for the function R is

$$R'(L) = B + AR + RD + RCR, \qquad L \geqslant 0 \tag{52}$$

and

$$R(0) = 0. \tag{53}$$

The original two-point boundary-value problem has been converted into a Cauchy problem. Equation (52) was obtained earlier in [4] and [5] by other modes of reasoning.

IV. Determination of Eigenvalues from the Invariant Imbedding Equations

Having obtained the invariant imbedding equations, it is now necessary to consider how the eigenvalues can be determined from them. By a physical interpretation of the variables $n(L)$, $p(L)$, $r(L)$, and $s(L)$ of Example 1, we note that the first two are the slope of the end $t = L$ of a column simply supported at $t = 0$ and having prescribed values of deflection and moment at $t = L$. The quantities $r(L)$ and $s(L)$ are the corresponding values of shearing force at the end $t = L$.

For a general length L, there is only the trivial solution to the homogeneous two-point boundary-value problem when $a = b = 0$. However, there are certain lengths L_1 for which a nontrivial solution can exist. If such a nontrivial solution does exist, then one or more of the quantities $n(L_1)$, $p(L_1)$, $r(L_1)$, and $s(L_1)$ is unbounded. Bear in mind that the parameter P is held fixed in our discussion.

For a given value of P, then, the task is to determine, analytically or computationally, the lengths $L = L_1$ for which any or all of the quantities $n(L)$, $p(L)$, $r(L)$, and $s(L)$ is unbounded. This is a relatively straightforward task from the computational point of view. The equations for n, p, r, and s are numerically integrated from $L = 0$ until one or more of the functions becomes exceedingly large. The value of L when this occurs is a lower bound on the first critical length, L_1. (Higher critical lengths are discussed later.) By repeating the calculation for various values of P, a curve of a lower bound on L_1 vs P is obtained. The same curve is also regarded as approximately giving the critical value P_1 as a function of length L, i.e., the compressive force at which a column of length L buckles.

In practice, during the course of computation, as the numerical values of n, p, r, and/or s become large, it is expedient to introduce the inverses of these functions,

n^{-1}, p^{-1}, etc. as new functions. The estimate of the critical lengths is given by the zero crossings. One can see, by switching back and forth between the functions and their inverses, how the numerical integration is allowed to proceed, yielding estimates of higher critical lengths. Thus, the higher eigenvalues are readily obtained.

The nonlinearity of the invariant imbedding equations poses no special difficulty, since experience has shown them to be numerically stable.

V. EXAMPLE SOLUTIONS

In this section, solutions to several cases of Examples 1 and 2 are presented. Both analytic and numerical solutions are illustrated. All solutions are compared to analytic solutions of the original form of the eigenvalue problem.

For the first solution, consider the special case of Example 1 where $k = 0$. It is shown in the appendix that the solution for s of Eq. (29) is

$$\sqrt{\frac{EI}{P}}\left[\tan^{-1}\left(\sqrt{\frac{EI}{P}}\,s\right) - \frac{\pi}{2}\right] = -L. \tag{54}$$

The variable s is seen to be infinite when

$$n\pi = \sqrt{\frac{P}{EI}}\,L_{cr} \qquad n = 1, 2, 3,\ldots,$$

or

$$P_{cr} = \frac{n^2\pi^2 EI}{L^2} \qquad n = 1, 2, 3,\ldots . \tag{55}$$

This is, of course, the expression for the classical Euler buckling load of a pinned-end column.

Now consider the special case of Example 2 where $k = 0$. For this case, the matrix differential equation (52) can be written in component form as

$$\begin{aligned} r_{11}' &= r_{21} - \frac{r_{11}r_{12}}{EI}, \\ r_{12}' &= -P + r_{22} - r_{11} - \frac{r_{12}^2}{EI}, \\ r_{21}' &= -\frac{r_{11}r_{22}}{EI}, \\ r_{22}' &= -r_{21} - \frac{r_{12}r_{22}}{EI}, \end{aligned} \tag{56}$$

with the initial conditions

$$r_{11}(0) = r_{12}(0) = r_{21}(0) = r_{22}(0) = 0. \tag{57}$$

Inspection of Eq. (56) with the associated initial conditions shows that

$$r_{21} = r_{22} = r_{11} = 0$$
$$r_{12}' = -P - \frac{r_{12}^2}{EI}. \tag{58}$$

From this last equation and the initial condition $r_{12}(0) = 0$ we find that

$$\sqrt{\frac{EI}{P}} \tan^{-1}\left(\frac{r_{12}}{\sqrt{PEI}}\right) = -L. \tag{59}$$

The critical lengths are then obtained when r_{12} is infinite, or

$$P_{cr} = \frac{(2n+1)\,\pi^2 EI}{4L^2} \qquad n = 0, 1, 2, \ldots. \tag{60}$$

This is identical to the Euler load for this problem as obtained by the classical eigenvalue approach.

Since the integration of an initial-value problem of a system of ordinary differential equations is a straightforward task for a digital computer, the program for determining the eigenvalues is quite simple. The storage requirements are minimal and no iterative procedures are required. A small computer can thus be used for solving eigenvalue problems. To illustrate this fact and to demonstrate the validity of the numerical technique, several problems have been solved on an IBM 1620 computer. The results of these calculations are shown in Table I.

TABLE I

Comparison of Results of Numerical Determinations of Critical Lengths

	Parameters			Method	
Example	P	EI	k	Analytic	Numerical
1[a]	1	1	0	3.1415926	3.1415936
1[b]	1	1	0	6.2831853	6.2831869
2	2	1	1	1.18962	1.18965

[a] First critical length.
[b] Second critical length.

In obtaining these results, two equivalent techniques are employed. The first is that directly indicated by the invariant imbedding formulation. The Riccati equations resulting from the invariant imbedding [e.g., Eqs. (29) and (52)] are numerically integrated using the appropriate initial conditions. In the examples shown, a fourth-order Runge-Kutta scheme is used. If the initial conditions are infinite, a large number (e.g., 10^3) is used in place of the infinite condition. Integration is allowed to proceed until the dependent variables begin to grow rapidly. At this point, the integration step size is reduced and the integration is continued. This process is continued until an overflow condition is reached in any of the variables. The value of the independent variable when the overflow occurs is the critical length of the column.

The necessity of working with "infinite" quantities is computationally unsatisfactory, and an alternate method is therefore proposed. If at any time in the course of the computations, any of the variables becomes large, the Riccati equations are subject to a transformation of the form

$$\text{New Variable} = (\text{Old Variable})^{-1}, \tag{61}$$

a technique employed by G. M. Wing [6]. In Example 1, where infinite initial conditions are required, this transformation is introduced at the start of the integration and the appropriate values for the derivatives at $L = 0$ are obtained by use of L'Hospital's rule.

If the second technique is used, the critical lengths are identified by the disappearance of the transformed variables. Both techniques are used in solving the example problems. The second technique is preferable in that overflow problems are eliminated and second and higher critical lengths can readily be obtained.

As illustrated by the table, the accuracy of the invariant imbedding method is very good. Greater accuracy can be obtained simply by taking smaller integration steps. Experience has shown that the invariant imbedding equations are numerically stable, thus simplifying the calculation.

VI. Conclusions

In the previous sections of this paper, a method for finding eigenvalues through the use of invariant imbedding has been developed. The great advantage of this approach is that the eigenvalue problem is converted into an initial-value problem. This initial-value form is readily integrated numerically. While the examples shown in this paper have treated the parameters as constants (i.e., k and EI), it is a simple matter to modify the equations to include the variable parameters.

Appendix: Solution of a Special Case of Equation (29)

In the special case of $k = 0$, Eqs. (29) and (30) can be written

$$\begin{aligned} n' &= -(n^2 + rp), \\ p' &= 1 - p(n + s), \\ r' &= -r(n + s), \\ s' &= -\frac{P}{EI} - (pr + s^2), \end{aligned} \tag{A-1}$$

$$\begin{aligned} n(0) &= +\infty, & r(0) &= 0, \\ p(0) &= 0, & s(0) &= +\infty. \end{aligned} \tag{A-2}$$

Observing that

$$r = 0 \tag{A–3}$$

is a possible solution, the above equations become

$$n' = -n^2, \tag{A-4a}$$

$$p' = 1 - p(n + s), \tag{A-4b}$$

$$s' = -\frac{P}{EI} - s^2. \tag{A-4c}$$

The first of these has the solution

$$n = \frac{1}{L + C}. \tag{A-5}$$

Enforcing the initial condition, it is seen that

$$n = \frac{1}{L}. \tag{A-6}$$

By separating variables in Eq. (A–4c) and integrating, we see that

$$\sqrt{\frac{EI}{P}} \tan^{-1}\left(\sqrt{\frac{EI}{P}}\, s\right) = -L + \text{Constant.} \tag{A-7}$$

Enforcing the initial condition we finally obtain

$$\tan^{-1}\left(\sqrt{\frac{EI}{P}}\, s\right) - \frac{\pi}{2} = -\sqrt{\frac{P}{EI}}\, L. \tag{A-8}$$

Using Eqs. (A–6) and (A–8), Eq. (A–4b) is seen to be a linear first-order equation. The solution of this equation can thus be obtained. Since the critical buckling length can be obtained from Eq. (A–8) alone, and since the solution for p will not add any insight into the problem, this solution will not be presented here.

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